Presentation Notes on: Further Properties of Subdifferentials and The Situation in $\mathbb{R}^n$

valentinpi

SEMINAR ON CONVEX ANALYSIS
Freie Universität Berlin
Summer Term 2023

July 25, 2023

Recap

Recap

Let X, Y be locally convex topological vector spaces, $n \in \mathbb{N}_{\geq 1}$ and $f \colon X \to \overline{\mathbb{R}}$.

Recap

Let X, Y be locally convex topological vector spaces, $n \in \mathbb{N}_{\geq 1}$ and $f\colon X \to \overline{\mathbb{R}}$.

Recall:

Effective domain, epigraph of a function.

Recap

Let X, Y be locally convex topological vector spaces, $n \in \mathbb{N}_{\geq 1}$ and $f\colon X \to \overline{\mathbb{R}}$.
Recall:

- Effective domain, epigraph of a function.
- Convexity of sets and functions.

Recap

Let X, Y be locally convex topological vector spaces, $n \in \mathbb{N}_{\geq 1}$ and $f\colon X \to \overline{\mathbb{R}}$.
Recall:

- Effective domain, epigraph of a function.
- Convexity of sets and functions.
- f is proper, if $-\infty < f \neq \infty$.

Recap

Let X, Y be locally convex topological vector spaces, $n \in \mathbb{N}_{\geq 1}$ and $f \colon X \to \overline{\mathbb{R}}$.
Recall:

- Effective domain, epigraph of a function.
- Convexity of sets and functions.
- f is proper, if $-\infty < f \neq \infty$.
- $x^* \in X^*$ is a subgradient of f at $x \in X$, if $f(z - x) \geq f(x) + \langle x^*, z - x \rangle \; \forall \, z \in X$.

Recap

Let X, Y be locally convex topological vector spaces, $n \in \mathbb{N}_{\geq 1}$ and $f\colon X \to \overline{\mathbb{R}}$.

Recall:

- Effective domain, epigraph of a function.
- Convexity of sets and functions.
- f is proper, if $-\infty < f \neq \infty$.
- $x^* \in X^*$ is a subgradient of f at $x \in X$, if $f(z - x) \geq f(x) + \langle x^*, z - x\rangle \; \forall\, z \in X$.
- The set of all subgradients is the subdifferential $\partial f(x)$.

Recap

Let X, Y be locally convex topological vector spaces, $n \in \mathbb{N}_{\geq 1}$ and $f \colon X \to \overline{\mathbb{R}}$.

Recall:

- Effective domain, epigraph of a function.
- Convexity of sets and functions.
- f is proper, if $-\infty < f \neq \infty$.
- $x^* \in X^*$ is a subgradient of f at $x \in X$, if $f(z - x) \geq f(x) + \langle x^*, z - x \rangle \; \forall \, z \in X$.
- The set of all subgradients is the subdifferential $\partial f(x)$.

So far: Definition, characterization, Moreau-Rockefellar.

Recap

Let X, Y be locally convex topological vector spaces, $n \in \mathbb{N}_{\geq 1}$ and $f\colon X \to \overline{\mathbb{R}}$.

Recall:

- Effective domain, epigraph of a function.
- Convexity of sets and functions.
- f is proper, if $-\infty < f \neq \infty$.
- $x^* \in X^*$ is a subgradient of f at $x \in X$, if $f(z - x) \geq f(x) + \langle x^*, z - x \rangle \; \forall z \in X$.
- The set of all subgradients is the subdifferential $\partial f(x)$.

So far: Definition, characterization, Moreau-Rockefellar.
Today: More properties, some for $X = \mathbb{R}^n$.

Recap: Important General Characterizations

Theorem 1 ([1, pp. 170-171])

Let f be proper and convex. Then the following statements are equivalent.

(i) f is bounded on a neighborhood of $x \in X$.

(ii) f is continuous at x.

Recap: Important General Characterizations

Theorem 2 ([1, pp. 193-199])

Let f be convex. Then the following statements are equivalent.

(i) $x^* \in \partial f(x)$.

(ii) $f(z) \geq f(x) + \langle x^*, z - x \rangle \, \forall \, z \in X$.

(iii) $f(x) + f^*(x^*) = \langle x^*, x \rangle$

(iv) $f'(x, y) = \sup_{x^* \in \partial f(x)} \langle x^*, y \rangle \geq \langle x^*, y \rangle \, \forall \, y \in X$, *i.e.* $x^* \in \partial f'(x; 0) = \partial f(x) = dom(f'(x; \cdot)^*)$, *if f is additionally proper.*

Recap: Important General Characterizations

Theorem 3 (Infimum Form [1, pp. 194-195])

Let f be proper and convex. Then $f'(x;\cdot)$ is well defined for any $x \in dom(f)$, convex, proper and for any $y \in X$

$$f'(x;y) = \inf_{\lambda \in \mathbb{R}_{>0}} \frac{f(x+\lambda y) - f(x)}{\lambda}$$

A Chain Rule

A Chain Rule

Theorem 4 ([1, p. 201])

Let $\Lambda \in L(X, Y)$ *and* $f\colon Y \to \overline{\mathbb{R}}$ *be a function.*

A Chain Rule

Theorem 4 ([1, p. 201])

Let $\Lambda \in L(X, Y)$ and $f\colon Y \to \overline{\mathbb{R}}$ be a function.

(i) For any $x \in X$, we have

$$\partial(f\Lambda)(x) \supseteq \Lambda^* \partial f(\Lambda x)$$

A Chain Rule

Theorem 4 ([1, p. 201])

Let $\Lambda \in L(X, Y)$ *and* $f\colon Y \to \overline{\mathbb{R}}$ *be a function.*

(i) For any $x \in X$*, we have*

$$\partial(f\Lambda)(x) \supseteq \Lambda^* \partial f(\Lambda x)$$

(ii) If f *is convex and proper on* X*, as well as continuous at a point in* $Im(\Lambda)$*, then for any* $x \in X$

$$\partial(f\Lambda)(x) = \Lambda^* \partial f(\Lambda x)$$

A Chain Rule

Theorem 4 ([1, p. 201])

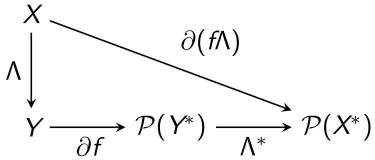
Let $\Lambda \in L(X, Y)$ *and* $f\colon Y \to \overline{\mathbb{R}}$ *be a function.*

(i) For any $x \in X$*, we have*

$$\partial(f\Lambda)(x) \supseteq \Lambda^* \partial f(\Lambda x)$$

(ii) If f *is convex and proper on* X*, as well as continuous at a point in* $Im(\Lambda)$*, then for any* $x \in X$

$$\partial(f\Lambda)(x) = \Lambda^* \partial f(\Lambda x)$$



A Chain Rule: Helper Theorems

A Chain Rule: Helper Theorems

Theorem 5 ([1, pp. 195-196])

Let $f\colon X \to \overline{\mathbb{R}}$ *be a proper, convex function, which is continuous on* $\emptyset \neq U \subseteq X$ *and* $x \in X$ *be fixed.*

(i) If $|f'(x;\bar{x})| < \infty$ *for* $\bar{x} \in X$ *with* $x + \bar{x} \in U$*, then* $f'(x;\cdot)$ *is continuous on* $K_{U \setminus \{x\}} \setminus \{0\}$ *or* $K_{U \setminus \{x\}}$*.*

(ii) If f *is continuous at* x*, then* $f'(x;\cdot)$ *is finite and continuous on* X*.*

A Chain Rule: Helper Theorems

Theorem 5 ([1, pp. 195-196])

Let $f\colon X \to \overline{\mathbb{R}}$ *be a proper, convex function, which is continuous on* $\emptyset \neq U \subseteq X$ *and* $x \in X$ *be fixed.*

(i) If $|f'(x;\bar{x})| < \infty$ *for* $\bar{x} \in X$ *with* $x + \bar{x} \in U$*, then* $f'(x;\cdot)$ *is continuous on* $K_{U\setminus\{x\}} \setminus \{0\}$ *or* $K_{U\setminus\{x\}}$.

(ii) If f *is continuous at* x*, then* $f'(x;\cdot)$ *is finite and continuous on* X.

Theorem 6 ([1, p. 179, p. 183])

Let $\Lambda \in L(X,Y)$ *and* $f\colon Y \to \overline{\mathbb{R}}$ *be a convex function, continuous at a point in* $\mathrm{Im}(\Lambda)$*. Then* $(f\Lambda)^* = \Lambda^* f^*$ *and for each* $x^* \in dom((f\Lambda)^*)$*, there is a* $y^* \in Y^*$ *with* $x^* = \Lambda^* y^*$ *and* $(f\Lambda)^*(x^*) = f^*(y^*)$.

The Supremum Function of Convex Functions is Convex

The Supremum Function of Convex Functions is Convex

Lemma 7

Let $\{f_s \colon X \to \overline{\mathbb{R}}\}_{s \in S}$*, with* S *an index set, be a family of convex functions and* $\hat{f} \colon X \to \overline{\mathbb{R}}, x \mapsto \sup_{s \in S} f(x)$*. Then* $\hat{f}$ *is convex.*

Supremum Function Subdifferential

Supremum Function Subdifferential

Theorem 8 ([1, pp. 201-204])

Let S be a compact topological space and $f\colon S \times X \to \overline{\mathbb{R}}$, s.t. for any $(s, x) \in S \times X$, $f|_{\{s\}\times X}$ is convex and proper, and that $f|_{S\times\{x\}}$ is upper semicontinuous. Let $f_s := f|_{\{s\}\times X}$ and set

$$\hat{f}\colon X \to \overline{\mathbb{R}}, x \mapsto \sup_{s\in S} f_s(x) \text{ and } S_0\colon X \to \mathcal{P}(S), x \mapsto \{s \in S \mid f_s(x) = \hat{f}(x)\}.$$

Supremum Function Subdifferential

Theorem 8 ([1, pp. 201-204])

Let S be a compact topological space and $f\colon S \times X \to \overline{\mathbb{R}}$, s.t. for any $(s,x) \in S \times X$, $f|_{\{s\}\times X}$ is convex and proper, and that $f|_{S\times\{x\}}$ is upper semicontinuous. Let $f_s := f|_{\{s\}\times X}$ and set

$$\hat{f}\colon X \to \overline{\mathbb{R}}, x \mapsto \sup_{s\in S} f_s(x) \text{ and } S_0\colon X \to \mathcal{P}(S), x \mapsto \{s \in S \mid f_s(x) = \hat{f}(x)\}.$$

(i) For any $x \in X$

$$\overline{conv}\left(\bigcup_{s\in S_0(x)} \partial f_s(x)\right) \subseteq \partial \hat{f}(x)$$

Supremum Function Subdifferential

Theorem 8 ([1, pp. 201-204])

Let S be a compact topological space and $f\colon S \times X \to \overline{\mathbb{R}}$, s.t. for any $(s,x) \in S \times X$, $f|_{\{s\}\times X}$ is convex and proper, and that $f|_{S\times\{x\}}$ is upper semicontinuous. Let $f_s := f|_{\{s\}\times X}$ and set

$$\hat{f}\colon X \to \overline{\mathbb{R}}, x \mapsto \sup_{s\in S} f_s(x) \text{ and } S_0\colon X \to \mathcal{P}(S), x \mapsto \{s \in S \mid f_s(x) = \hat{f}(x)\}.$$

(i) For any $x \in X$

$$\overline{conv}\left(\bigcup_{s\in S_0(x)} \partial f_s(x)\right) \subseteq \partial\hat{f}(x)$$

(ii) If for all $s \in S$, f_s is continuous at a point $x_0 \in X$, then

$$\overline{conv}\left(\bigcup_{s\in S_0(x_0)} \partial f_s(x_0)\right) = \partial\hat{f}(x_0)$$

Supremum Function Subdifferential: Helper Separation Theorem

Supremum Function Subdifferential: Helper Separation Theorem

Theorem 9 ([1, pp. 164-165])

Let $A \subseteq X$ be closed, convex and let $x \notin A$. Then there exists a functional $x^* \in X^*$, s.t. $\langle x^*, y\rangle \leq \langle x^*, x\rangle - \varepsilon$ for a fixed $\varepsilon \in \mathbb{R}_{>0}$ and any $y \in A$, strongly separating A and $\{x\}$.

Subdifferentials in $\mathbb{R}^n$: Existence

Subdifferentials in $\mathbb{R}^n$: Existence

Theorem 10 ([1, p. 204])

Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper and convex. Then f is subdifferentiable in $ri(dom(f))$.

Subdifferentials in $\mathbb{R}^n$: Existence

Theorem 10 ([1, p. 204])

Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *be proper and convex. Then* f *is subdifferentiable in* $ri(dom(f))$.

One helper theorem is needed.

Subdifferentials in $\mathbb{R}^n$: Existence

Theorem 10 ([1, p. 204])
Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper and convex. Then f is subdifferentiable in $ri(dom(f))$.

One helper theorem is needed.

Lemma 11 ([1, p. 188])
Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex and proper.

(i) f is continuous with respect to $aff(dom(f))$ on $ri(dom(f))$.

(ii) f^* is proper.

Subdifferentials in $\mathbb{R}^n$: Representation

Subdifferentials in $\mathbb{R}^n$: Representation

Theorem 12 ([1, pp. 204-205])

Let $X, S, f, f_s, \hat{f}, S_0, x_0$ for $X = \mathbb{R}^n$ and any $s \in S$ be defined as in the setting of Theorem 8 (ii). Then every $y \in \partial\hat{f}(x_0)$ can be represented as a convex combination of form

$$y = \sum_{i=1}^{r} \alpha_i y_i$$

with $r \in \mathbb{N}$, $1 \leq r \leq n+1$, $\sum_{i=1}^{r} \alpha_i = 1$ and $s_i \in S_0(x_0)$, $(\alpha_i, y_i) \in \mathbb{R}_{>0} \times \partial f_{s_i}(x_0)$ for any $i \in \mathbb{N}$, $1 \leq i \leq r$.

Subdifferentials in $\mathbb{R}^n$: Representation - Helper Theorems

Subdifferentials in $\mathbb{R}^n$: Representation - Helper Theorems

Lemma 13 ([1, pp. 185-186])

Let $A \subseteq \mathbb{R}^n$ be bounded and closed. Then $conv(A) = \overline{conv}(A)$.

Subdifferentials in $\mathbb{R}^n$: Representation - Helper Theorems

Lemma 13 ([1, pp. 185-186])

Let $A \subseteq \mathbb{R}^n$ be bounded and closed. Then $conv(A) = \overline{conv}(A)$.

Lemma 14 ([1, p. 199])

For a proper convex function $f\colon X \to \overline{\mathbb{R}}$, which is continuous at a point x_0, $\partial f(x_0)$ is non-empty, weakly* bounded and weakly* compact.

Subdifferentials in $\mathbb{R}^n$: Representation - Helper Theorems

Lemma 13 ([1, pp. 185-186])

Let $A \subseteq \mathbb{R}^n$ be bounded and closed. Then $conv(A) = \overline{conv}(A)$.

Lemma 14 ([1, p. 199])

For a proper convex function $f\colon X \to \overline{\mathbb{R}}$, which is continuous at a point x_0, $\partial f(x_0)$ is non-empty, weakly* bounded and weakly* compact.

Lemma 15

Let $X, S, f, f_s, \hat{f}, S_0, x_0$ be defined as in the setting of Theorem 8 (ii). Then $S_0(x_0)$ is compact.

References

[1] A.D. Ioffe and V.M. Tihomirov, *Theory of Extremal Problems*, ISBN: 9780444851673.