Seminar on Convex Analysis
Freie Universität Berlin
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Presentation Handout on: Further Properties of Subdifferentials and The Situation in $\mathbb{R}^n$

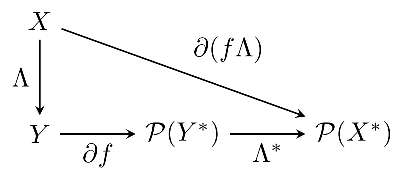
Theorem 1 (**A Chain Rule** [1, p. 201])**.** Let $\Lambda \in L(X, Y)$ and $f\colon Y \to \overline{\mathbb{R}}$ be a function.

(i) For any $x \in X$, we have

$$\partial(f\Lambda)(x) \supseteq \Lambda^* \partial f(\Lambda x) \tag{1}$$

(ii) If f is convex and proper on X, as well as continuous at a point in $\mathrm{Im}(\Lambda)$, then for any $x \in X$

$$\partial(f\Lambda)(x) = \Lambda^* \partial f(\Lambda x) \tag{2}$$



Theorem 2 (**Supremum Functions** [1, pp. 201-204])**.** Let S be a compact topological space and $f\colon S \times X \to \overline{\mathbb{R}}$, s.t. for any $(s, x) \in S \times X$, $f|_{\{s\}\times X}$ is convex and proper, and that $f|_{S\times\{x\}}$ is upper semicontinuous, where we each omit the respective fixed argument. Let $f_s \coloneqq f|_{\{s\}\times X}$ and set

$$\hat{f}\colon X \to \overline{\mathbb{R}}, x \mapsto \sup_{s\in S} f_s(x) \text{ and } S_0\colon X \to \mathcal{P}(S), x \mapsto \{s \in S \mid f_s(x) = \hat{f}(x)\}. \tag{3}$$

In both of the following statements, the closures are taken wrt. the weak*-topology of X^*.

(i) For any $x \in X$

$$\overline{\mathrm{conv}}\left(\bigcup_{s\in S_0(x)} \partial f_s(x)\right) \subseteq \partial\hat{f}(x) \tag{4}$$

(ii) If for all $s \in S$, f_s is continuous at a point $x_0 \in X$, then

$$\overline{\mathrm{conv}}\left(\bigcup_{s\in S_0(x_0)} \partial f_s(x_0)\right) = \partial\hat{f}(x_0) \tag{5}$$

Let $n \in \mathbb{N}_{\geq 1}$.

Theorem 3 ([1, p. 204])**.** Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper and convex. Then f is subdifferentiable in $\mathrm{ri}(\mathrm{dom}(f))$.

Theorem 4 ([1, pp. 204-205])**.** Let $X, S, f, f_s, \hat{f}, S_0, x_0$ for $X = \mathbb{R}^n$ and any $s \in S$ be defined as in the setting of Theorem 2 (ii). Then every $y \in \partial\hat{f}(x_0)$ can be represented as a convex combination of form

$$y = \sum_{i=1}^{r} \alpha_i y_i \tag{6}$$

with $r \in \mathbb{N}$, $1 \leq r \leq n+1$, $\sum_{i=1}^{r} \alpha_i = 1$ and $s_i \in S_0(x_0)$, $(\alpha_i, y_i) \in \mathbb{R}_{>0} \times \partial f_{s_i}(x_0)$ for any $i \in \mathbb{N}$, $1 \leq i \leq r$.

References

[1] A.D. Ioffe and V.M. Tihomirov, *Theory of Extremal Problems*, ISBN: 9780444851673.