Seminar on Measure and Integration Theory	valentinpi
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Presentation Handout on: Measure Decomposition Theorems
For this presentation, we used mainly [1, pp. 71-76, pp. 113-118], but also [1] as a whole and [2, 3].

1 Decomposition Theorems for Measures

Throughout this document, assume that $(X, \mathfrak{M})$ is a measurable space.

Definition 1.1. Let $\mu, \nu\colon \mathfrak{M} \to [0,\infty]$ be two measures. We define the following three set functions:

$$\nu_{ac}\colon \mathfrak{M} \to [0,\infty], E \mapsto \max\left\{\int_E u\,d\mu \,\middle|\, u\colon X \to [0,\infty] \text{ measurable} \wedge \int_{E'} u\,d\mu \le \nu(E')\,\forall\,\mathfrak{M} \ni E' \subseteq E\right\} \tag{1}$$

$$\nu_s\colon \mathfrak{M} \to [0,\infty], E \mapsto \max\{\nu(E') \mid \mathfrak{M} \ni E' \subseteq E, \mu(E') = 0\} \tag{2}$$

$$\nu_d\colon \mathfrak{M} \to [0,\infty], E \mapsto \max\{\nu(E') \mid \mathfrak{M} \ni E' \subseteq E, \mu(E'') = \infty\,\forall\,\mathfrak{M} \in E'' \subseteq E', \nu(E'') > 0\} \tag{3}$$

Theorem 1 (Lebesgue Decomposition Theorem)**.** Let $\mu, \nu\colon \mathfrak{M} \to [0,\infty]$ be two measures and μ σ-finite.

(i) Then

$$\nu = \nu_{ac} + \nu_s. \tag{4}$$

(ii) If ν is σ-finite, then $\nu_s \perp \mu$ and the decomposition is unique.

Theorem 2 (De Giorgis Theorem)**.** Let $\mu, \nu\colon \mathfrak{M} \to [0,\infty]$ be two measures. Then

$$\nu = \nu_{ac} + \nu_s + \nu_d. \tag{5}$$

2 Decomposition Theorems for Signed Measures

Let $\lambda\colon \mathfrak{M} \to [-\infty,\infty]$ be a signed measure.

Lemma 2.1. Let $E \in \mathfrak{M}$ with $\lambda(E) \in (0,\infty)$. Then there exists a positive $\mathfrak{M} \ni F \subseteq E$.

Theorem 2 (Hahn Decomposition Theorem)**.** Any measurable space $(X, \mathfrak{M})$ can be decomposed as $X = X^+ \cup X^-$, where $X^+ \subseteq X$ is positive and $X^- \subseteq X$ is negative.

Theorem 3 (Jordan Decomposition Theorem)**.** There exists a unique pair (λ^+, λ^-) of measures with $\lambda^+ \perp \lambda^-$, one being finite and $\lambda = \lambda^+ - \lambda^-$.

Theorem 4 (Signed Lebesgue Decomposition Theorem)**.** Let $\lambda\colon \mathfrak{M} \to [-\infty,\infty]$ be a signed measure and $\mu\colon \mathfrak{M} \to [0,\infty]$ be a σ-finite measure.

(i) There are signed measures $\lambda_{ac}, \lambda_s\colon X \to [-\infty,\infty]$ and a measurable function $u\colon X \to [-\infty,\infty]$ with

$$\lambda = \lambda_{ac} + \lambda_s, \lambda_{ac} \ll \mu \text{ and } \lambda_{ac}(\cdot) = \int_\cdot u\,d\mu. \tag{6}$$

(ii) If λ is σ-finite, then $\lambda_s \perp \mu$ and the decomposition is unique.

Decomposition	Summary
Hewitt-Yosida [1, pp. 8-9]	$\mu = \mu_p + \mu_c$
Atomic [1, pp. 13-16]	$\mu = \mu_1 + \mu_2$
Lebesgue	$\nu = \nu_{ac} + \nu_s$
De Giorgi	$\nu = \nu_{ac} + \nu_s + \nu_d$
Hahn	$X = X^+ \cup X^-$
Jordan	$\lambda = \lambda^+ - \lambda^-$
Signed Lebesgue	$\lambda = \lambda_{ac} + \lambda_s$

References

[1] I. Fonseca and G. Leoni, *Modern methods in the calculus of variations*, ISBN: 978-0-387-35784-3.
[2] S. J. Axler, *Measure, integration & real analysis*, ISBN: 978-3-030-33142-9.
[3] J. Elstrodt, Ed., *Maß- und Integrationstheorie*, ISBN: 978-3-540-89727-9.