Presentation Notes on: Measure Decomposition Theorems

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Recap

Let $(X, \mathfrak{M})$ be a measurable space throughout.

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So far:

Theorem 1 (Radon-Nikodym I [1, pp. 56-59])

Let $\mu, \nu \colon \mathfrak{M} \to [0, \infty]$ be two measures with μ σ-finite and $\nu \ll \mu$. Then there is a measurable $u \colon X \to [0, \infty]$ with

$$\nu(\cdot) = \int_{\cdot} u \, d\mu, \tag{1}$$

which is unique up to a set of μ-measure zero.

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Today: More decomposition theorems.

Recap II

Definition 2 ([1, p. 55])

Let $\mu, \nu \colon \mathfrak{M} \to [0, \infty]$ be two measures.

(i) ν is said to be *absolutely continuous* wrt. μ, $\nu \ll \mu$, if for any $E \in \mathfrak{M}$, $\mu(E) = 0 \to \nu(E) = 0$.

(ii) μ, ν are said to be *mutually singular*, $\nu \perp \mu$, if there are disjoint $X_\mu, X_\nu \in \mathfrak{M}$ with $X = X_\mu \cup X_\nu$ and $\mu(E) = \mu(E \cap X_\mu)$, as well as $\nu(E) = \nu(E \cap X_\nu)$ for any $E \in \mathfrak{M}$.

(iii) ν is said to be *diffuse* wrt. μ, if for any $E \in \mathfrak{M}$, $\mu(E) < \infty \to \nu(E) = 0$.

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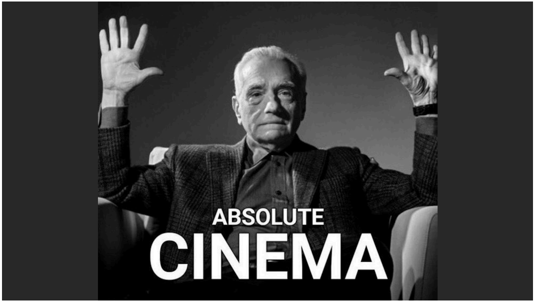

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Auxiliary Functions I

Definition 3

Let $\mu, \nu \colon \mathfrak{M} \to [0, \infty]$ be two measures. We define the following three set functions:

$$\nu_{ac} \colon \mathfrak{M} \to [0, \infty], E \mapsto \max\left\{ \int_E u \, d\mu \,\middle|\, u \colon X \to [0, \infty] \text{ measurable} \wedge \int_{E'} u \, d\mu \leq \nu(E') \,\forall\, \mathfrak{M} \ni E' \subseteq E \right\} \tag{2}$$

$$\nu_s \colon \mathfrak{M} \to [0, \infty], E \mapsto \max\{\nu(E') \mid \mathfrak{M} \ni E' \subseteq E, \mu(E') = 0\} \tag{3}$$

$$\nu_d \colon \mathfrak{M} \to [0, \infty], E \mapsto \max\{\nu(E') \mid \mathfrak{M} \ni E' \subseteq E, \mu(E'') = \infty \,\forall\, \mathfrak{M} \in E'' \subseteq E', \nu(E'') > 0\} \tag{4}$$

Auxiliary Functions II

Lemma 4

The following statements hold.

(i) ν_{ac}, ν_s *and* ν_d *are well-defined and measures.*

(ii) $\nu_{ac} \ll \mu$.

(iii) If ν_s *is* σ*-finite, then there exists some* $X_s \in \mathfrak{M}$*, s.t.*

$$\mu(X_s) = 0 = \nu_d(X_s) \text{ and } \nu_s(E) = \nu(E \cap X_s) \tag{5}$$

for all $E \in \mathfrak{M}$*. Also,* $\nu_s \perp \mu$ *and* $\nu_s \perp \nu_d$.

(iv) ν_d *is diffuse wrt.* μ.

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Lemma 5 ([1, pp. 13-14])

Let $\mu\colon \mathfrak{M} \to [0, \infty]$ *be a measure,* $\mathfrak{N} \subseteq \mathfrak{M}$ *be closed under finite unions and* $\emptyset \in \mathfrak{N}$*. Then*

$$\nu\colon \mathfrak{M} \to [0, \infty], E \mapsto \max\{\mu(E \cap F) \mid F \in \mathfrak{N}\} \tag{6}$$

is well-defined and a measure.

Lebesgue Decomposition

Theorem 6 (Lebesgue Decomposition Theorem)

Let $\mu, \nu \colon \mathfrak{M} \to [0, \infty]$ *be two measures and* μ σ*-finite.*

(i) Then

$$\nu = \nu_{ac} + \nu_s. \tag{7}$$

(ii) If ν *is* σ*-finite, then* $\nu_s \perp \mu$ *and the decomposition is unique.*

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Lemma 7 ([1, p. 56, 64])

Let $\mu, \nu \colon \mathfrak{M} \to [0, \infty]$ *be measures with* μ σ*-finite and* $\nu \ll \mu$*. Then* $\nu = \nu_{ac}$*.*

De Giorgis Theorem

$\rightsquigarrow$ Lebesgue for non-σ-finite μ.

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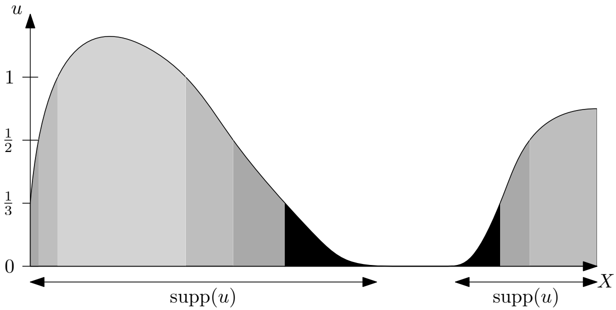
Theorem 8 (De Giorgis Theorem)

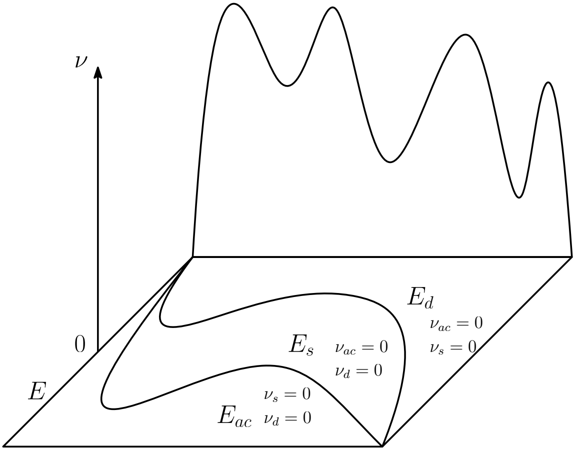
Let $\mu, \nu \colon \mathfrak{M} \to [0, \infty]$ *be two measures. Then*

$$\nu = \nu_{ac} + \nu_s + \nu_d. \tag{8}$$

De Giorgis Theorem: Proof Strategy I



De Giorgis Theorem: Proof Strategy II



Signed Measures

Definition 9

A function $\lambda\colon \mathfrak{M} \to [-\infty, \infty]$ is called a *signed measure*, if:

(i) $\lambda(\emptyset) = 0$

(ii) $|\{-\infty, \infty\} \cap \mathrm{im}(\lambda)| \leq 1$

(iii) For any mutually disjoint family $\{E_n \subseteq \mathfrak{M}\}_{n\in\mathbb{N}_{\geq 1}}$, we have $\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n)$.

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Lemma 10

A set function $\lambda \colon \mathfrak{M} \to [-\infty, \infty]$ is a signed measure, iff it satisfies the following:

(i) $|\{-\infty, \infty\} \cap \mathit{im}(\lambda)| \leq 1$

(ii) $\lambda(E \cup F) = \lambda(E) + \lambda(F)$ *for disjoint* $E, F \in \mathfrak{M}$.

(iii) For any increasing sequence $\{E_n \subseteq \mathfrak{M}\}_{n \in \mathbb{N}_{\geq 1}}$, *we have* $\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \lambda(E_n)$.

Positive Sets

Definition 11

A set $E \in \mathfrak{M}$ is called *positive*, if $\lambda(F) \geq 0$, and resp. *negative*, if $\lambda(F) \leq 0$, for all $\mathfrak{M} \ni F \subseteq E$.

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Lemma 12
Let $E \in \mathfrak{M}$ with $\lambda(E) \in (0, \infty)$. Then there exists a positive $\mathfrak{M} \ni F \subseteq E$.

Hahn Decomposition

Theorem 13 (Hahn Decomposition Theorem)

Any measurable space $(X, \mathfrak{M})$ *can be decomposed as* $X = X^+ \cup X^-$*, where* $X^+ \subseteq X$ *is positive and* $X^- \subseteq X$ *is negative.*

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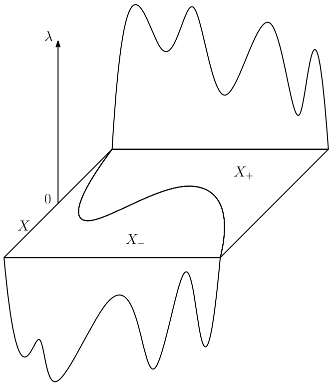
Theorem 13 (Hahn Decomposition Theorem)

Any measurable space $(X, \mathfrak{M})$ can be decomposed as $X = X^+ \cup X^-$, where $X^+ \subseteq X$ is positive and $X^- \subseteq X$ is negative.



Jordan Decomposition

Theorem 14 (Jordan Decomposition Theorem)

There exists a unique pair (λ^+, λ^-) *of measures with* $\lambda^+ \perp \lambda^-$*, one being finite and* $\lambda = \lambda^+ - \lambda^-$*.*

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$\rightsquigarrow$ Lebesgue integral definition.

Signed Lebesgue Decomposition

„Culmination Theorem".

Theorem 15 (Signed Lebesgue Decomposition Theorem)

Let $\lambda\colon \mathfrak{M} \to [-\infty, \infty]$ *be a signed measure and* $\mu\colon \mathfrak{M} \to [0, \infty]$ *be a* σ*-finite measure.*

(i) There are signed measures $\lambda_{ac}, \lambda_s\colon X \to [-\infty, \infty]$ *and a measurable function* $u\colon X \to [-\infty, \infty]$ *with*

$$\lambda = \lambda_{ac} + \lambda_s, \tag{9}$$

$\lambda_{ac} \ll \mu$ *and*

$$\lambda_{ac}(\cdot) = \int_{\cdot} u \, d\mu. \tag{10}$$

(ii) If λ *is* σ*-finite, then* $\lambda_s \perp \mu$ *and the decomposition is unique.*

Summary

Decomposition	Summary
Hewitt-Yosida [1, pp. 8-9]	$\mu = \mu_p + \mu_c$
Atomic [1, pp. 13-16]	$\mu = \mu_1 + \mu_2$
Lebesgue	$\nu = \nu_{ac} + \nu_s$
De Giorgi	$\nu = \nu_{ac} + \nu_s + \nu_d$
Hahn	$X = X^+ \cup X^-$
Jordan	$\lambda = \lambda^+ - \lambda^-$
Signed Lebesgue	$\lambda = \lambda_{ac} + \lambda_s$

References

The main source of this presentation is [1], additionally the books [2] and [3] were used.

[1] I. Fonseca und G. Leoni, *Modern methods in the calculus of variations*, ISBN: 978-0-387-35784-3.

[2] S. J. Axler, *Measure, integration & real analysis*, ISBN: 978-3-030-33142-9.

[3] J. Elstrodt, Hrsg., *Maß- und Integrationstheorie*, ISBN: 978-3-540-89727-9.