Seminar on Online Algorithms
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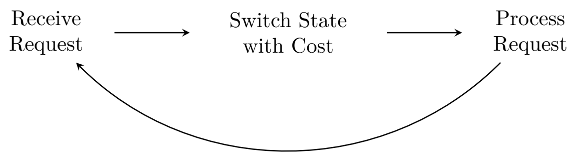
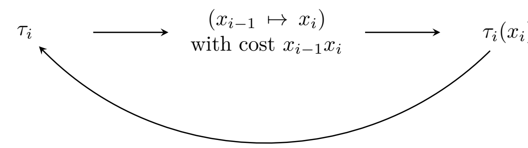
**Handout on: Metrical Task Systems and the Work Function Algorithm**

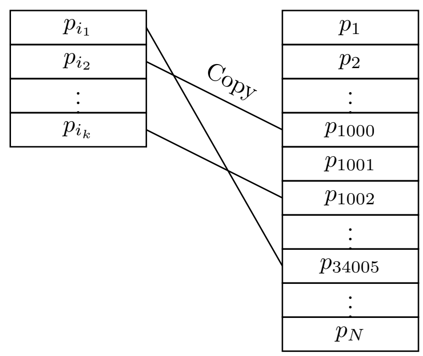
**Definition 1** ([1, pp. 75-76])**.** A *Metrical Task System* (MTS) is a tuple $((\mathcal{M}, d), \mathcal{T})$, where $(\mathcal{M}, d)$ is a finite metric space [2, pp. 3-4] of cardinality $N := |\mathcal{M}|$, the *set of states*, and $\mathcal{T} \subseteq \left(\mathbb{R}_{\geq 0}^{\infty}\right)^N$ is the *set of tasks*. If $d(x, y) = 1$ for any $x, y \in \mathcal{M}$ with $x \neq y$, then the MTS is called *uniform*.

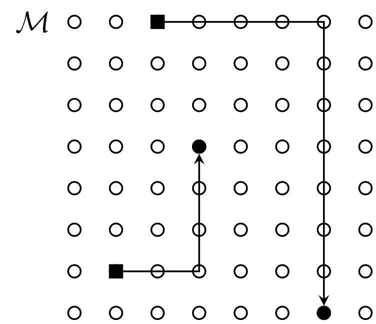



*Examples:* The Ice Cream Problem [1, pp. 74-75], the Paging Problem [3, p. 124] and the $k$-Server Problem [1, pp. 87-88].

| | $V$ | $S$ |
|---|---|---|
| $V_M$ | 1 | 4 |
| $S_M$ | 2 | 2 |




**Definition 2** ([1, p. 76])**.** Let $x \in \mathcal{M}^n$ be a one-indexed sequence of states. Set

$$\text{(1)} \qquad \operatorname{cost}(x_0, \tau, x) := \sum_{i=1}^{n} x_{i-1}x_i + \tau_i(x_i), \qquad \operatorname{opt}(x_0, \tau) := \min_{x \in \mathcal{M}^n} \operatorname{cost}(x_0, \tau, x)$$

Let further $\mathcal{T}^* := \bigcup_{m=1}^{\infty} \mathcal{T}^m$. A (deterministic) *online strategy/online algorithm* for $\mathcal{M}$ is a map $\mathcal{A} \colon \mathcal{M} \times \mathcal{T}^* \to \mathcal{M}$. We further set the cost of execution by $\mathcal{A}$ as

$$\text{(2)} \qquad \operatorname{cost}_{\mathcal{A}}(x_0, \tau) := \operatorname{cost}(x_0, \tau, \mathcal{A}(x_0, \tau_1)\mathcal{A}(x_0, \tau_1\tau_2)...\mathcal{A}(x_0, \tau_1\tau_2...\tau_n))$$

$\mathcal{A}$ is called *$c$-competitive with initial function* $\alpha \colon \mathcal{M} \to \mathcal{R}$, where $c \in \mathbb{R}$, if for any $x_0 \in \mathcal{M}$ and $\tau \in \mathcal{T}^*$, we have

$$\text{(3)} \qquad \operatorname{cost}_{\mathcal{A}}(x_0, \tau) \leq c \operatorname{opt}(x_0, \tau) + \alpha(x_0)$$

Furthermore, the value $\operatorname{argmin}_{c \in \mathbb{R}}(\mathcal{A}$ is $c$-competitive), if it exists, is called the *competitive-ratio* of $\mathcal{A}$.

**Definition 3** (Work Functions)**.** The function

$$\text{(4)} \qquad \omega \colon \mathcal{M} \to \mathbb{R}, x \mapsto \min_{(x_1, \ldots, x_{n-1}) \in \mathcal{M}^{n-1}} \operatorname{cost}(x_0, \tau, (x_0, x_1, ..., x_{n-1}, x))$$

is called the *work function* for $x_0$ and $\tau$.

**Dynamic Program for Computing the Work Function** Let $\omega_i$ for any $i \in \mathbb{N}$, $0 \le i \le n$, denote the work function of $x_0$ and $(\tau_1, ..., \tau_i)$ computing the lowest costs for the first $i$ tasks. Then the following dynamic program allows the computation of $\omega_0, ..., \omega_n$.

$$\omega_0(x) = x_0 x \tag{5}$$

$$\omega_{i+1}(x) = \min_{x' \in \mathcal{M}} \omega_i(x') + \tau_{i+1}(x') + x'x \tag{6}$$

**The Work Function Algorithm (WFA)** Let $i \in \mathbb{N}$, $1 \le i \le n-1$ and suppose the states $x_0, x_1, ..., x_i$ have already been computed. Then the WFA chooses the next state by

$$x_{i+1} \in \arg\min_{x \in \mathcal{M}} \omega_{i+1}(x) + x_i x \tag{†}$$

$$\text{s.t. } \omega_{i+1}(x_{i+1}) = \omega_i(x_{i+1}) + \tau_{i+1}(x_{i+1}) \tag{‡}$$

**Theorem 1** ([3, pp. 132-133])**.** The WFA is well-defined.

**Theorem 2** ([3, pp. 133-134])**.** The WFA is $(2N-1)$-competitive.

**Theorem 3** ([3, pp. 128-132])**.** The competitive ratio of any deterministic online algorithm for general MTS is lower bounded by $(2N-1)$.

**Conjecture 1** ([3, pp. 152-153])**.** There exists a $k$-competitive deterministic online algorithm for the $k$-server algorithm over any metric space.

| Problem | $c_{\text{WFA}}$ | $c_{\text{Better}}$ |
|---|---|---|
| Ice Cream | 3 | 7/6 (claim in exercise, [1, pp. 80-82]) |
| Paging | $2\binom{N}{k} - 1$ | $k$ (tight, [1, pp. 54-56]) |
| $k$-Server MTS WFA | $2N^k - 1$ | See below |
| 2-Server WFA | 2 (tight, [1, pp. 87-89, p. 94]) | None |
| $k$-Server WFA, $k \ge 3$ | $2k-1$ ([1, pp. 92-93]) | Open |
| $k$-Server WFA, $\lvert\mathcal{M}\rvert = k+2$ | $k$ (tight, [1, pp. 87-89, pp. 94-95]) | None |

## References


[1] G. Woeginger, *Online Algorithms*, ISBN: 978-3-540-64917-4.
[2] O. Forster, *Analysis 2*, ISBN: 978-3-658-19410-9.
[3] A. Borodin, *Online Computation and Competitive Analysis*, ISBN: 978-0-521-56392-5.