Computing roots with interval nestings

valentinpi

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Abstract

Computing roots of positive reals using a calculator is quite easy, but implementing it? In this paper, I discuss a quick way (not as in *efficient*) of approximating roots of any real number. Then I present a small implementation in the Python programming language. Much of the work here is taken from [1].

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Contents

1 Introduction

1.1 Notation

I denote the set of natural numbers with $\mathbb{N}$, the set of real numbers with $\mathbb{R}$ and the set of positive reals with $\mathbb{R}_+$. I also assume basic knowledge in set theory and notation, mathematical proofs and notation and fields.

There are a few things which need to be known about the field $\mathbb{R}$, but I will try to keep this paper short and only mention the most important ones for the most important proofs.

2 Construction of roots

2.1 Completeness of $\mathbb{R}$

The real numbers $\mathbb{R}$ are structured in three ways:

- Field structure through it's axioms and all derivable calculation rules
- Ordering of real numbers with the property of positivity
- Completeness

We will take a look at the latter. With the rational numbers, we are not able to describe every point in a unit line

$$\{ x \in \mathbb{R} \mid 0 \le x \le 1 \}$$

One famous example for this is the reciprocal of the golden ratio. For the irrationality proof, consider reading [1]. While the rational numbers therefore are, in a sense, incomplete, the real numbers do not have this issue.
There are multiple ways of introducing the completeness of the field $\mathbb{R}$. [1] introduces this concept, for one, using so-called interval nestings.

Definition 1. Let $a, b \in \mathbb{R}$ with $a < b$. We define the *closed interval*

$$[a; b] := \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

with the *boundaries* a, b. The number $b - a =: |[a; b]|$ is called *length* of the interval. Closed intervals are also commonly known as *compact*.

This definition should be well-known. Now to interval nestings.

Definition 2. An *interval nesting* is a series of compact intervals $\mathrm{I}_1, \mathrm{I}_2, ...$, short (I_n), that satisfy the following two properties

(I.1) $\mathrm{I}_{n+1} \supset \mathrm{I}_n$

(I.2) $\forall \varepsilon > 0 \, \exists \, n \in \mathbb{N} \colon |\mathrm{I}_n| < \varepsilon$

The latter property (I.2) can be interpreted as that the intervals can get arbitrarily small. We can define interval nestings using induction.
The completeness of $\mathbb{R}$ is founded on the following theorem.

Theorem (Principle of Interval Nesting)**.** For every interval nesting (I_n) in the $\mathbb{R}$, there exists a real number s, which is included in every interval.

The theorem itself does not exclude multiple numbers s, but the following proof will rule that out.

Theorem 1. The number s is unique.

Proof. Let's say two numbers $s \neq t$ are included in every interval. Without loss of generality, say $s < t$. Then every interval is of length $\geq t - s$, which contradicts (I.2). ↯ □

2.2 Existence of roots

With the introduction of interval nestings, we can now prove a theorem that directly constructs roots. Some prerequisites first, which we will use.

Theorem 2 (Archimedes Axiom)**.** For every $x \in \mathbb{R}$, there is a $n \in \mathbb{N}$, so that

$$n > x$$

The statement of this axiom is clear. It goes along with two more axioms in the ordering of the real numbers, which is not mentioned here.

Lemma 1*.* Let $a > b$. Then

$$\frac{1}{a} < \frac{1}{b}$$

for the reciprocals.

This is a well known theorem. We will not prove it here, but it is important for the following proofs.

Lemma 2 (Bernoulli's Inequality)*.* Let $x \in \mathbb{R}$ with $x \neq 0, x > -1$ and $n \in \mathbb{N}, n \geq 2$. Then the following holds

$$(1 + x)^n > 1 + nx$$

Proof by induction after n.
Base case) Let $n = 2$. Then

$$(1+x)^2 \geq 1+2x \Leftrightarrow x^2+2x+1 \geq 1+2x$$

which holds for every x specified.
$n \rightsquigarrow n+1$) Consider the following

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n \cdot (1+x) \\ &\geq (1+nx)\cdot(1+x) && \text{True for } < n+1 \\ &\geq 1+(n+1)x\end{aligned}$$

By the principle of induction, the statement is true. □

This lemma will be useful for the next one, which we will use later to prove (I.2) for constructed interval nestings.

Lemma 3. Let $0 < q < 1$. For every $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ so that the following holds

$$q^n < \varepsilon$$

Proof. Write $q' := q^{-1}$ and $\varepsilon' := \varepsilon^{-1}$. Since $q' > 1$ due to the ordering in $\mathbb{R}$, we can write $q' = 1 + x$ for $x := q' - 1$. Therefore we can use Bernoulli's Inequality above:

$$(1+x)^n > 1+nx \text{ for one } n \geq 2$$

Which n do we choose? We use the Archimedes Axiom. There is a natural $n \geq 2$ such that $n > \frac{\varepsilon'}{x}$ and therefore

$$(q')^n > 1+nx > nx > \varepsilon'$$

Now we have $(q')^n > \varepsilon'$. If we take the reciprocals, then according to the lemma above, we get the theorem.

$$q^n < \varepsilon$$

□

Now for the main part.

Theorem 3. For every real number $x > 0$ and every $k \in \mathbb{N}$, there is one and only one real number $y > 0$ with $y^k = x$. We call it the kth root of x, in symbols $y = x^{\frac{1}{x}}, y = \sqrt[k]{x}$.

Proof. We consider the case $x > 1$, since for $x < 1$ we can make the transition $x' := 1/x$, which complies with the ordering of $\mathbb{R}$. And for $x = 1$ it holds that $y = 1$.

First, we construct an interval nesting (I_n) in the $\mathbb{R}_+$.
We begin with the properties our nesting should hold. For every interval $I_n = [a_n; b_n], n \in \mathbb{N}$, the following should hold

(1_n) $a_n^k \leq x \leq b_n^k$

(2_n) $|\mathrm{I}_n| = \left(\frac{1}{2}\right)^{n-1} \cdot |\mathrm{I}_1|$

So every interval interval I_{n+1} is half the length of the previous interval I_n.

Now for the construction we use induction. We declare the first interval and then, by the properties of the natural numbers, declare the next ones.
Let $\mathrm{I}_1 := [1; x]$. We verify that the properties (1_1) and (2_1) hold.
Now let n be arbitrary, but fixed, with $\mathrm{I}_n = [a_n; b_n]$ holding the properties (1_n) and (2_n).
We construct the next interval $n+1$ by cutting the nth one in half. Let $m := \frac{b_n - a_n}{2}$ be the center of the interval. We then define

$$\mathrm{I}_{n+1} := \begin{cases} [a_n; m] & m \geq x \\ [m; b_n] & m < x \end{cases}$$

Due to our construction, (1_{n+1}) and (2_{n+1}) both hold, since the newly constructed interval is in the bounds of I_n and therefore included. And because the new interval is exactly half the previous one.

We must now still prove that both (I.1) and (I.2) hold.

(I.1) See above.

(I.2) With (2_n) consider

$$\left(\frac{1}{2}\right)^{n-1=n'} < \varepsilon' = \varepsilon \cdot |\mathrm{I}_1|^{-1}$$

With the lemma we have proven above, there exists such an integer n' and therefore

$$\left(\frac{1}{2}\right)^{n-1} \cdot |\mathrm{I}_1| < \varepsilon$$

which satisfies the property.

Let y be the number that is included in all intervals (I_n). We will now show that $y^k = x$.

For that, we first prove that the intervals $([a_n^k; b_n^k])$ also form an interval nesting.

Since the first intervals all include y, these ones include y^k, which is in the bounds of the a_n^k, b_n^k. We also know, that all these intervals include the number x. Due to the uniqueness of the contained number, we derive that $x = y^k$ and therefore $y = \sqrt[k]{x}$.

□

3 Implementation

3.1 Algorithm design

3.2 Complexity considerations

References

[1] Konrad Königsberger. *Analysis 1*. Springer-Verlag, 2004.